

NONUNIFORM (μ, ν) -DICHOTOMIES AND LOCAL DYNAMICS OF DIFFERENCE EQUATIONS

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ABSTRACT. We obtain a local stable manifold theorem for perturbations of nonautonomous linear difference equations possessing a very general type of nonuniform dichotomy, possibly with different growth rates in the uniform and nonuniform parts. We note that we consider situations where the classical Lyapunov exponents can be zero. Additionally, we study how the manifolds decay along the orbit of a point as well as the behavior under perturbations and give examples of nonautonomous linear difference equations that admit the dichotomies considered.

1. INTRODUCTION

The main purpose of this paper is to discuss, in a Banach space X , the existence of stable manifolds for a general family of perturbations of nonautonomous linear difference equation

$$x_{m+1} = A_m x_m + f_m(x_m), \quad m \in \mathbb{N},$$

assuming that the perturbations $f_m: X \rightarrow X$ verify $f_m(0) = 0$,

$$\|f_m(u) - f_m(v)\| \leq c\|u - v\|(\|u\| + \|v\|)^q, \quad m \in \mathbb{N},$$

for some constants $c > 0$ and $q > 1$ and for each $u, v \in X$, and that the linear equation

$$x_{m+1} = A_m x_m, \quad m \in \mathbb{N},$$

admits a very general type of nonuniform dichotomy given by arbitrary rates of growth.

The notion of uniform exponential dichotomy was introduced by Perron in [11] and constitutes a very important tool in the theory of difference and differential equations, particularly in the study of invariant manifolds. In spite of being used in a wide range of situations, sometimes this notion is too demanding and it is of interest to consider more general kinds of hyperbolic behavior. A much more general type of dichotomy, allowing the rates of growth to vary along the trajectory of a point, is the notion of nonuniform exponential dichotomy that was introduced by Barreira and Valls in the context of nonautonomous differential equations in [4] and that was inspired both in Perron's classical notion of exponential dichotomy and in the notion of nonuniformly hyperbolic trajectory introduced by Pesin in [12, 13, 14]. In the context of difference equations, it was also introduced a notion of nonuniform exponential dichotomy in [3].

Date: January 20, 2013.

2010 Mathematics Subject Classification. 37D10, 34D09, 37D25.

Key words and phrases. Invariant manifolds, nonautonomous difference equations, nonuniform generalized dichotomies.

The study of stable manifolds in the nonuniform context has a long history, starting with a famous theorem on existence of stable manifolds for nonuniformly hyperbolic trajectories, in the finite dimensional setting, proved by Pesin [12]. In [16] Ruelle gave an alternative proof of this theorem based on the study of perturbations of products of matrices occurring in Oseledec's multiplicative ergodic theorem [10] and, inspired in the classical work of Hadamard, Pugh and Shub [15] proved the same result using graph transform techniques. In Hilbert spaces and under some compactness assumptions, Ruelle [17] obtained a version of the stable manifold theorem, following his approach in [16]. Versions of this theorem for transformations in Banach spaces, were established first by Mañé [9] under some compactness and invertibility assumptions and then by Thieullen [18] under weaker hypothesis.

Stable manifold were also obtained for perturbations of nonautonomous linear differential equations and for perturbations of nonautonomous linear difference equations, assuming respectively that the linear differential equation and linear difference equation admit a nonuniform exponential dichotomy. We refer the reader to the book [5], where the obtention of stable manifolds for perturbations of linear differential equations admitting the existence of nonuniform exponential dichotomies is discussed, and also to [3, 2, 1] for a related discussion in the context of difference equations.

Recently, invariant stable manifolds were obtained for perturbations of nonautonomous linear difference and differential equations, assuming the existence of nonuniform dichotomies that are not exponential. In particular, in the discrete time setting, assuming the existence of a some type of polynomial dichotomy for a nonautonomous linear difference equation, it was established in [6] the existence of local stable manifolds for a certain class of perturbations and, for a more restricted class, there were also obtained global stable manifolds.

Our result can be seen as a discrete counterpart of the results obtained in [7] for nonautonomous differential equations and we emphasize that the stable manifold theorem for perturbations of linear difference equations with nonuniform exponential dichotomies in [3] is included in our theorem as a very particular case and our result also includes as particular cases stable manifold theorems for polynomial dichotomies, as well as many other situations where the classical Lyapunov exponent is zero. We stress that, to the best of our knowledge, in the context of perturbations of nonautonomous linear difference equations that admit a non-exponential nonuniform dichotomy, our result is the first one addressing the existence of local stable manifolds for the general class of perturbations above. In particular, it is new even for nonuniform polynomial dichotomies (in [6] it was already considered the polynomial case but the type of nonuniform dichotomies considered there were different from the ones considered here). In the context of differential equations and under the existence of nonuniform polynomial dichotomies it were also obtained local and global stable manifolds in [8].

As mentioned, the type of dichotomies considered in this paper are very general, allowing different rates of growth for the uniform and the nonuniform parts and thus, to establish the existence of stable manifolds, we must assume conditions relating the rate of decay of the some balls in the stable spaces and the growth rates.

To highlight the generality of this concept of dichotomy, we discuss some families of new examples that verify the hypothesis in our main result. Additionally, we

obtain a upper bound for the decay of solutions along the stable manifolds and we study how the stable manifolds vary with the perturbations by giving bounds, in some suitable metric, on the distances between the functions whose graphs are the stable manifolds.

The content of the paper is the following: in Section 2 we introduce some notation, the main definitions and state the main theorem; next, in Section 3 we present some examples; then, in Section 4, we prove the main theorem; finally, in Section 5 we study how the manifolds obtained vary with the perturbations considered.

2. MAIN RESULT

We say that an increasing sequence $\mu = (\mu_n)_{n \in \mathbb{N}_0}$ is a *growth rate* if $\mu_0 \geq 1$ and $\lim_{n \rightarrow +\infty} \mu_n = +\infty$.

Let $\mu = (\mu_n)_{n \in \mathbb{N}_0}$ and $\nu = (\nu_n)_{n \in \mathbb{N}_0}$ be growth rates and let $B(X)$ be the space of bounded linear operators in a Banach space X . Given a sequence $(A_n)_{n \in \mathbb{N}}$ of invertible operators of $B(X)$ and putting

$$\mathcal{A}_{m,n} = \begin{cases} A_{m-1} \cdots A_n & \text{if } m > n, \\ \text{Id} & \text{if } m = n, \end{cases}$$

we say that the linear difference equation

$$x_{m+1} = A_m x_m, \quad m \in \mathbb{N} \quad (1)$$

admits a *nonuniform (μ, ν) -dichotomy* if there exist projections P_m , $m \in \mathbb{N}$, such that

$$P_m \mathcal{A}_{m,n} = \mathcal{A}_{m,n} P_n, \quad m, n \in \mathbb{N},$$

and constants $a < 0 \leq b$, $\varepsilon \geq 0$ and $D \geq 1$ such that for every $n \in \mathbb{N}$ and every $m \geq n$,

$$\|\mathcal{A}_{m,n} P_n\| \leq D \left(\frac{\mu_m}{\mu_{n-1}} \right)^a \nu_{n-1}^\varepsilon, \quad (2)$$

$$\|\mathcal{A}_{m,n}^{-1} Q_m\| \leq D \left(\frac{\mu_{m-1}}{\mu_n} \right)^{-b} \nu_{m-1}^\varepsilon, \quad (3)$$

where $Q_m = \text{Id} - P_m$ is the complementary projection. When $\varepsilon = 0$ we say that we have a *uniform μ -dichotomy* or simply a *μ -dichotomy*.

In these conditions we define, for each $n \in \mathbb{N}$, the linear subspaces $E_n = P_n(X)$ and $F_n = Q_n(X)$. As usual, we identify the vector spaces $E_n \times F_n$ and $E_n \oplus F_n$ as the same vector space.

We are going to address the problem of existence of stable manifolds of the difference equation

$$x_{m+1} = A_m x_m + f_m(x_m), \quad m \in \mathbb{N},$$

where $f_m : X \rightarrow X$ are perturbations for which there are constants $c > 0$ and $q > 1$ such that

$$f_m(0) = 0, \quad (4)$$

$$\|f_m(u) - f_m(v)\| \leq c \|u - v\| (\|u\| + \|v\|)^q \quad (5)$$

for every $m \in \mathbb{N}$ and every $u, v \in X$. Note that making $v = 0$ in (5) we have

$$\|f_m(u)\| \leq c \|u\|^{q+1} \quad (6)$$

for every $m \in \mathbb{N}$ and every $u \in X$.

Given $n \in \mathbb{N}$ and $v_n = (\xi, \eta) \in E_n \times F_n$, for each $m > n$ we write

$$v_m = \mathcal{F}_{m,n}(v_n) = \mathcal{F}_{m,n}(\xi, \eta) = (x_m, y_m) \in E_m \times F_m, \quad (7)$$

with

$$\mathcal{F}_{m,n} = \begin{cases} (A_{m-1} + f_{m-1}) \circ \cdots \circ (A_n + f_n) & \text{if } m > n, \\ \text{Id} & \text{if } m = n. \end{cases} \quad (8)$$

We denote by $B_n(r)$ the open ball of E_n centered at zero and with radius $r > 0$. Fix now $\delta > 0$ and let $\beta = (\beta_n)_{n \in \mathbb{N}}$ be a positive sequence. We denote by $\mathcal{X}_{\delta, \beta}$ the space of sequences $(\varphi_n)_{n \in \mathbb{N}}$ of continuous functions $\varphi_n : B_n(\delta\beta_n) \rightarrow F_n$ such that

$$\varphi_n(0) = 0 \quad (9)$$

$$\|\varphi_n(\xi) - \varphi_n(\bar{\xi})\| \leq \|\xi - \bar{\xi}\| \quad (10)$$

for every $\xi, \bar{\xi} \in B_n(\delta\beta_n)$ and every $n \in \mathbb{N}$. Given $(\varphi_n)_{n \in \mathbb{N}} \in \mathcal{X}_{\delta, \beta}$, for each $n \in \mathbb{N}$, we consider the graph

$$\mathcal{V}_{\varphi, n, \delta, \beta} = \{(\xi, \varphi_n(\xi)) : \xi \in B_n(\delta\beta_n)\}, \quad (11)$$

that we call *local stable manifold*.

We now state the result on the existence of local stable manifolds and its proof will be given in Section 4.

Theorem 1. *Given a Banach space X , let $f_m : X \rightarrow X$ be a sequence of functions satisfying (4) and (5) for some $c > 0$ and $q > 1$. Suppose equation (1) admits a nonuniform (μ, ν) -dichotomy for some growth rates μ and ν , $D \geq 1$, $a < 0 \leq b$ and $\varepsilon \geq 0$. Assume that*

$$\lim_{m \rightarrow +\infty} \mu_m^a \mu_{m-1}^{-b} \nu_{m-1}^\varepsilon = 0 \quad (12)$$

and that

$$\sum_{k=1}^{+\infty} \mu_k^{aq} \nu_k^\varepsilon \text{ is convergent.} \quad (13)$$

Define the sequences $\beta = (\beta_m)_{m \in \mathbb{N}}$ and $\tilde{\beta} = (\tilde{\beta}_m)_{m \in \mathbb{N}}$ by

$$\beta_m = \frac{\mu_{m-1}^a}{\nu_{m-1}^{\varepsilon(1+1/q)} \left(\sum_{k=m}^{+\infty} \mu_k^{aq} \nu_k^\varepsilon \right)^{1/q}} \quad \text{and} \quad \tilde{\beta}_m = \beta_m \nu_{m-1}^{-\varepsilon} \quad (14)$$

and suppose that there is a constant $K \geq 1$ such that

$$\frac{\mu_m^a \beta_m^{-1}}{\mu_{n-1}^a \beta_n^{-1}} \leq K \text{ for every } n \in \mathbb{N} \text{ and every } m \geq n. \quad (15)$$

Then, for every $C > D$, choosing $\delta > 0$ sufficiently small, there is a unique $\varphi \in \mathcal{X}_{\delta, \beta}$ such that

$$\mathcal{F}_{m,n}(\mathcal{V}_{\varphi, n, \delta/(CK), \tilde{\beta}}) \subseteq \mathcal{V}_{\varphi, m, \delta, \beta} \text{ for every } n \in \mathbb{N} \text{ and every } m \geq n. \quad (16)$$

where $\mathcal{V}_{\varphi, n, \delta/(CK), \tilde{\beta}}$ and $\mathcal{V}_{\varphi, m, \delta, \beta}$ are given by (11). Furthermore, given $n \in \mathbb{N}$, we have

$$\|\mathcal{F}_{m,n}(\xi, \varphi_n(\xi)) - \mathcal{F}_{m,n}(\bar{\xi}, \varphi_n(\bar{\xi}))\| \leq 2C \left(\frac{\mu_m}{\mu_{n-1}} \right)^a \nu_{n-1}^\varepsilon \|\xi - \bar{\xi}\|. \quad (17)$$

for every $m \geq n$ and $\xi, \bar{\xi} \in B_n(\delta\tilde{\beta}_n/(CK))$.

3. EXAMPLES

In this section we will illustrate our main result with some examples. Firstly, we will give examples of linear difference equations admitting nonuniform (μ, ν) -dichotomies for any growth rates μ and ν . Secondly, we show that the nonuniform exponential result obtained in [3] is a particular case of our theorem and finally we highlight two new settings where our main theorem can be applied.

Example 1. Given $a < 0 \leq b$ and $\varepsilon \geq 0$, let $(A_n)_{n \in \mathbb{N}}$ be the sequence of bounded linear operators $A_n : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ given by the diagonal matrices

$$A_n = \begin{bmatrix} \left(\frac{\mu_{n+1}}{\mu_{n-1}} \right)^a \left(\frac{\nu_n^{\cos(n\pi)-1}}{\nu_{n-1}^{\cos((n-1)\pi)-1}} \right)^{\varepsilon/2} & 0 \\ 0 & \left(\frac{\mu_{n+1}}{\mu_n} \right)^b \left(\frac{\nu_n^{\cos(n\pi)-1}}{\nu_{n-1}^{\cos((n-1)\pi)-1}} \right)^{\varepsilon/2} \end{bmatrix}$$

where $\mu = (\mu_n)_{n \in \mathbb{N}_0}$ and $\nu = (\nu_n)_{n \in \mathbb{N}_0}$ are two growth rates. Then

$$\mathcal{A}_{m,n} = \begin{bmatrix} \left(\frac{\mu_{m-1}\mu_m}{\mu_{n-1}\mu_n} \right)^a \left(\frac{\nu_{m-1}^{\cos((m-1)\pi)-1}}{\nu_{n-1}^{\cos((n-1)\pi)-1}} \right)^{\varepsilon/2} & 0 \\ 0 & \left(\frac{\mu_m}{\mu_n} \right)^b \left(\frac{\nu_{m-1}^{\cos((m-1)\pi)-1}}{\nu_{n-1}^{\cos((n-1)\pi)-1}} \right)^{\varepsilon/2} \end{bmatrix}$$

and considering the projections given by $P_n(x, y) = (x, 0)$ and $Q_n(x, y) = (0, y)$ we have

$$\|\mathcal{A}_{m,n}P_n\| = \left(\frac{\mu_{m-1}}{\mu_n} \right)^a \left(\frac{\mu_m}{\mu_{n-1}} \right)^a \left(\frac{\nu_{m-1}^{\cos((m-1)\pi)-1}}{\nu_{n-1}^{\cos((n-1)\pi)-1}} \right)^{\varepsilon/2}$$

and

$$\|\mathcal{A}_{m,n}^{-1}Q_m\| = \left(\frac{\mu_m}{\mu_{m-1}} \right)^{-b} \left(\frac{\mu_{m-1}}{\mu_n} \right)^{-b} \left(\frac{\nu_{m-1}^{\cos((m-1)\pi)-1}}{\nu_{n-1}^{\cos((n-1)\pi)-1}} \right)^{-\varepsilon/2}$$

and this implies

$$\|\mathcal{A}_{m,n}P_n\| \leq \left(\frac{\mu_m}{\mu_{n-1}} \right)^a \nu_{n-1}^\varepsilon \quad \text{and} \quad \|\mathcal{A}_{m,n}^{-1}Q_m\| \leq \left(\frac{\mu_{m-1}}{\mu_n} \right)^{-b} \nu_{m-1}^\varepsilon.$$

This example shows that for every growth rates μ and ν we have a nonuniform (μ, ν) -dichotomy.

Moreover, if m is even, n is odd and μ_m/μ_{m-1} is bounded by a constant λ then

$$\lambda^{-b} \left(\frac{\mu_{m-1}}{\mu_n} \right)^{-b} \nu_{m-1}^\varepsilon \leq \|\mathcal{A}_{m,n}^{-1}Q_m\| \leq \left(\frac{\mu_{m-1}}{\mu_n} \right)^{-b} \nu_{m-1}^\varepsilon$$

and this shows that the nonuniform part of the dichotomy can not be removed.

Example 2. With $\mu_n = \nu_n = e^n$ we get the local stable manifold theorem obtained by Barreira and Valls in [3]. Here, condition (12) becomes $a + \varepsilon < b$, condition (13) becomes $aq + \varepsilon < 0$ and since

$$\beta_m = e^{-a+\varepsilon(1+1/q)} (1 - e^{aq+\varepsilon})^{1/q} e^{-\varepsilon(1+2/q)m}$$

and

$$\frac{\mu_m^a \beta_m^{-1}}{\mu_{n-1}^a \beta_n^{-1}} = e^a e^{(a+\varepsilon(1+2/q))(m-n)},$$

condition (15) becomes $a + \varepsilon(1 + 2/q) \leq 0$, that is the condition $a + \beta \leq 0$ in [3]. Note that condition $a + \varepsilon(1 + 2/q) \leq 0$ implies $a + \varepsilon < b$ and $aq + \varepsilon < 0$, although the first implication seems to have been unnoticed in [3].

Example 3. We will now consider the polynomial case, i.e., $\mu_n = \nu_n = 1 + n$. For these rates, condition (12) becomes $a + \varepsilon < b$ and condition (13) becomes $aq + \varepsilon + 1 < 0$. Since

$$\int_m^{+\infty} (1+t)^{aq+\varepsilon} dt \leq \sum_{k=m}^{+\infty} \mu_k^{aq} \nu_k^\varepsilon = \sum_{k=m}^{+\infty} (1+k)^{aq+\varepsilon} \leq \int_{m-1}^{+\infty} (1+t)^{aq+\varepsilon} dt,$$

we obtain the estimates

$$\frac{1}{|aq + \varepsilon + 1|} (1+m)^{aq+\varepsilon+1} \leq \sum_{k=m}^{+\infty} (1+k)^{aq+\varepsilon} \leq \frac{1}{|aq + \varepsilon + 1| 2^{aq+\varepsilon+1}} (1+m)^{aq+\varepsilon+1}$$

and this implies

$$\beta_m \leq \frac{|aq + \varepsilon + 1|^{1/q}}{2^{a-\varepsilon(1+1/q)}} (1+m)^{-\varepsilon(1+2/q)-1/q}$$

and

$$\beta_m \geq 2^{a+\varepsilon/q+1/q} |aq + \varepsilon + 1|^{1/q} (1+m)^{-\varepsilon(1+2/q)-1/q}.$$

Hence

$$\frac{\mu_m^a \beta_m^{-1}}{\mu_{n-1}^a \beta_n^{-1}} \leq 2^{-2a+\varepsilon-1/q} \left(\frac{1+m}{1+n} \right)^{a+\varepsilon(1+2/q)+1/q}$$

and to have condition (15) we need to have $a + \varepsilon(1 + 2/q) + 1/q \leq 0$. Therefore, taking into account that $a + \varepsilon(1 + 2/q) + 1/q \leq 0$ implies $a + \varepsilon < b$ and also implies $aq + \varepsilon + 1 < 0$ when $\varepsilon > 0$, if $a + \varepsilon(1 + 2/q) + 1/q \leq 0$ and $\varepsilon > 0$ we have a local stable manifold theorem. If $aq + 1 < 0$ and $\varepsilon = 0$ we also have a local stable manifold theorem.

Example 4. In this example we will consider a nonuniform dichotomy with the following growth rates

$$\mu_n = (1+n)(1+\log(1+n))^\lambda \quad \text{and} \quad \nu_n = 1+\log(1+n),$$

with $\lambda \geq 0$. Then condition (12) is satisfied for every $a < 0 \leq b$ and every $\varepsilon \geq 0$. The series in (13) becomes

$$\sum_{k=1}^{\infty} \mu_k^{aq} \nu_k^\varepsilon = \sum_{k=1}^{\infty} (1+k)^{aq} (1+\log(1+k))^{\lambda aq+\varepsilon}$$

and is convergent if $aq < -1$ or if $aq = -1$ and $\varepsilon - \lambda < -1$.

If $aq < -1$, there are positive constants θ_1 and θ_2 such that

$$\sum_{k=m}^{\infty} \mu_k^{aq} \nu_k^\varepsilon \geq \theta_1 (1+m)^{aq+1} (1+\log(1+m))^{\lambda aq+\varepsilon}$$

and

$$\sum_{k=m}^{\infty} \mu_k^{aq} \nu_k^\varepsilon \leq \theta_2 (1+m)^{aq+1} (1+\log(1+m))^{\lambda aq+\varepsilon}$$

for every $m \in \mathbb{N}$ and this implies that

$$\beta_m \leq \frac{\theta_1^{-1/q}}{2^a(1+\log 2)^{\lambda a - \varepsilon(1+1/q)}} (1+m)^{-1/q} (1+\log(1+m))^{-\varepsilon(1+2/q)}$$

and

$$\beta_m \geq \theta_2^{-1/q} (1+m)^{-1/q} (1+\log(1+m))^{-\varepsilon(1+2/q)}$$

for every $m \in \mathbb{N}$. Hence

$$\frac{\mu_m^a \beta_m^{-1}}{\mu_{n-1}^a \beta_n^{-1}} \leq A \left(\frac{m+1}{n+1} \right)^{a+1/q} \left(\frac{1+\log(1+m)}{1+\log(1+n)} \right)^{\lambda a + \varepsilon(1+2/q)}$$

with

$$A = \frac{\theta_2^{1/q}}{\theta_1^{1/q} 2^a (1+\log 2)^{\lambda a - \varepsilon(1+1/q)}}$$

and since $aq < -1$ condition (15) is always satisfied. Therefore if $aq < -1$ we have a local stable manifold theorem for every nonuniform dichotomy with these growth rates.

When $aq = -1$ and $\varepsilon - \lambda < -1$, there are positive constants θ_3 and θ_4 such that

$$\theta_3 (1+\log(1+m))^{-\lambda + \varepsilon + 1} \leq \sum_{k=m}^{+\infty} \mu_k^{-1} \nu_k^\varepsilon \leq \theta_4 (1+\log(1+m))^{-\lambda + \varepsilon + 1}$$

for every $m \in \mathbb{N}$. This estimates imply that

$$\beta_m \leq \frac{\theta_3^{-1/q}}{2^a(1+\log 2)^{\lambda a - \varepsilon(1+1/q)}} (1+m)^{-1/q} (1+\log(1+m))^{-\varepsilon(1+2/q)-1/q}$$

and

$$\beta_m \geq \theta_4^{-1/q} (1+m)^{-1/q} (1+\log(1+m))^{-\varepsilon(1+2/q)-1/q}$$

for every $m \in \mathbb{N}$. Hence

$$\frac{\mu_m^a \beta_m^{-1}}{\mu_{n-1}^a \beta_n^{-1}} \leq \frac{\theta_4^{1/q}}{\theta_3^{1/q} 2^a (1+\log 2)^{-\lambda/q - \varepsilon(1+1/q)}} \left(\frac{1+\log(1+m)}{1+\log(1+n)} \right)^{(1-\lambda)/q + \varepsilon(1+2/q)}$$

for every $m \in \mathbb{N}$ and condition (15) is satisfied if $(1-\lambda)/q + \varepsilon(1+2/q) \leq 0$. Since $(1-\lambda)/q + \varepsilon(1+2/q) \leq 0$ and $\varepsilon > 0$ imply $\varepsilon - \lambda < -1$, if $aq = -1$ and $(1-\lambda)/q + \varepsilon(1+2/q) \leq 0$ and $\varepsilon > 0$ we have a local stable manifold theorem for nonuniform dichotomies with these rates. If $aq = -1$, $\varepsilon = 0$ and $\lambda > 1$ we also have a local stable manifold theorem.

4. PROOF OF THEOREM 1

Given $n \in \mathbb{N}$ and $v_n = (\xi, \eta) \in E_n \times F_n$, using (7), it follows that for each $m > n$, the trajectory $(v_m)_{m>n}$ satisfies the following equations

$$x_m = \mathcal{A}_{m,n} \xi + \sum_{k=n}^{m-1} \mathcal{A}_{m,k+1} P_{k+1} f_k(x_k, y_k), \quad (18)$$

$$y_m = \mathcal{A}_{m,n} \eta + \sum_{k=n}^{m-1} \mathcal{A}_{m,k+1} Q_{k+1} f_k(x_k, y_k). \quad (19)$$

In view of the forward invariance mentioned in (16), each trajectory of (8) starting in $\mathcal{V}_{\varphi,n,\delta/(CK),\tilde{\beta}}$ must be in $\mathcal{V}_{\varphi,m,\delta,\beta}$ for every $m \geq n$, and thus the equations (18) and (19) can be written in the form

$$x_m = \mathcal{A}_{m,n}\xi + \sum_{k=n}^{m-1} \mathcal{A}_{m,k+1} P_{k+1} f_k(x_k, \varphi_k(x_k)), \quad (20)$$

$$\varphi_m(x_m) = \mathcal{A}_{m,n}\varphi_n(\xi) + \sum_{k=n}^{m-1} \mathcal{A}_{m,k+1} Q_{k+1} f_k(x_k, \varphi_k(x_k)). \quad (21)$$

To prove that equations (20) and (21) have solutions we will use Banach fixed point theorem in some suitable complete metric spaces.

In $\mathcal{X}_{\delta,\beta}$ we define a metric by

$$\|\varphi - \psi\|' = \sup \left\{ \frac{\|\varphi_n(\xi) - \psi_n(\xi)\|}{\|\xi\|} : n \in \mathbb{N} \text{ and } \xi \in B_n(\delta\beta_n) \setminus \{0\} \right\}. \quad (22)$$

for each $\varphi = (\varphi_n)_{n \in \mathbb{N}}$, $\psi = (\psi_n)_{n \in \mathbb{N}} \in \mathcal{X}_{\delta,\beta}$. It is easy to see that $\mathcal{X}_{\delta,\beta}$ is a complete metric space with the metric defined by (22).

We also need to consider the space $\mathcal{X}_{\delta,\beta}^*$ of sequences $\varphi = (\varphi_n)_{n \in \mathbb{N}}$ with $\varphi_n: E_n \rightarrow F_n$ such that the sequence $(\varphi_n|_{B_n(\delta\beta_n)})_{n \in \mathbb{N}}$ is in $\mathcal{X}_{\delta,\beta}$ and, for each $n \in \mathbb{N}$,

$$\varphi_n(\xi) = \varphi_n \left(\frac{\delta\beta_n \xi}{\|\xi\|} \right) \text{ whenever } \xi \notin B_n(\delta\beta_n).$$

There is a one-to-one correspondence between sequences in $\mathcal{X}_{\delta,\beta}$ and in $\mathcal{X}_{\delta,\beta}^*$ because for each sequence of functions $\varphi = (\varphi_n)_{n \in \mathbb{N}} \in \mathcal{X}_{\delta,\beta}$ there is a unique extension $\tilde{\varphi} = (\tilde{\varphi}_n)_{n \in \mathbb{N}}$ such that each $\tilde{\varphi}_n$ is a Lipschitz extension of φ_n to $\overline{B_n(\delta\beta_n)}$. This one-to-one correspondence allows to define a metric in $\mathcal{X}_{\delta,\beta}^*$. For every $\varphi = (\varphi_n)_{n \in \mathbb{N}}$, $\psi = (\psi_n)_{n \in \mathbb{N}} \in \mathcal{X}_{\delta,\beta}^*$, we define this metric by

$$\|\varphi - \psi\|' = \|\overline{\varphi} - \overline{\psi}\|'$$

where $\overline{\varphi} = (\varphi_n|_{B_n(\delta\beta_n)})_{n \in \mathbb{N}}$, $\overline{\psi} = (\psi_n|_{B_n(\delta\beta_n)})_{n \in \mathbb{N}}$ and the right hand side is the metric defined by (22). It is easy to see that with this metric $\mathcal{X}_{\delta,\beta}^*$ is a complete metric space.

Furthermore, given $\varphi = (\varphi_n)_{n \in \mathbb{N}}$, $\psi = (\psi_n)_{n \in \mathbb{N}} \in \mathcal{X}_{\delta,\beta}^*$, one can easily verify that

$$\|\varphi_n(\xi) - \varphi_n(\bar{\xi})\| \leq 2\|\xi - \bar{\xi}\| \quad (23)$$

$$\|\varphi_n(\xi) - \psi_n(\xi)\| \leq \|\varphi - \psi\|' \|\xi\| \quad (24)$$

for every $n \in \mathbb{N}$ and every $\xi, \bar{\xi} \in E_n$.

Let $\mathcal{B} = \mathcal{B}_{n,\delta,\beta}$ be the space of all sequences $x = (x_m)_{m \geq n}$ of functions

$$x_m: B_n(\delta\beta_n) \rightarrow E_m$$

such that

$$x_n(\xi) = \xi, \quad x_m(0) = 0 \quad (25)$$

$$\|x_m(\xi) - x_m(\bar{\xi})\| \leq C \left(\frac{\mu_m}{\mu_{n-1}} \right)^a \nu_{n-1}^\varepsilon \|\xi - \bar{\xi}\| \quad (26)$$

for every $m \geq n$ and every $\xi, \bar{\xi} \in B_n(\delta\beta_n)$. Making $\bar{\xi} = 0$ in (26) we obtain the following estimates

$$\|x_m(\xi)\| \leq C \left(\frac{\mu_m}{\mu_{n-1}} \right)^a \nu_{n-1}^\varepsilon \|\xi\| \leq C\delta \left(\frac{\mu_m}{\mu_{n-1}} \right)^a \nu_{n-1}^\varepsilon \beta_n \quad (27)$$

for every $m \geq n$ and every $\xi \in B_n(\delta\beta_n)$. In $\mathcal{B}_{n,\delta,\beta}$ we define a metric by

$$\|x - y\|'' = \sup \left\{ \frac{\|x_m(\xi) - y_m(\xi)\|}{\|\xi\|} \left(\frac{\mu_m}{\mu_{n-1}} \right)^{-a} \nu_{n-1}^{-\varepsilon} : m \geq n, \xi \in B_n(\delta\beta_n) \right\} \quad (28)$$

for every $x, y \in \mathcal{B}_{n,\delta,\beta}$. It is easy to see that with this metric $\mathcal{B}_{n,\delta,\beta}$ is a complete metric space.

Lemma 1. *Given $\delta > 0$ sufficiently small, for each $\varphi \in \mathcal{X}_{\delta,\beta}^*$ and $n \in \mathbb{N}$ there exists a unique sequence $x = x^\varphi \in \mathcal{B}_{n,\delta,\beta}$ satisfying the equation (20) for every $m \geq n$ and $\xi \in B_n(\delta\beta_n)$. Moreover, choosing $\delta > 0$ sufficiently small, we have*

$$\|x^\varphi - x^\psi\|'' \leq \frac{C}{3} \nu_{n-1}^{-\varepsilon} \|\varphi - \psi\|' \quad (29)$$

for each $\varphi, \psi \in \mathcal{X}_{\delta,\beta}^*$.

Proof. Given $\varphi \in \mathcal{X}_{\delta,\beta}^*$, we define an operator $J = J_\varphi$ in $\mathcal{B}_{n,\delta,\beta}$ by

$$(Jx)_m(\xi) = \begin{cases} \xi & \text{if } m = n, \\ \mathcal{A}_{m,n}\xi + \sum_{k=n}^{m-1} \mathcal{A}_{m,k+1} P_{k+1} f_k(x_k(\xi), \varphi_k(x_k(\xi))) & \text{if } m > n. \end{cases} \quad (30)$$

One can easily verify from (25), (9) and (4) that $(Jx)_m(0) = 0$ for every $m \geq n$.

Let $x \in \mathcal{B}_{n,\delta,\beta}$ and, for every $k \geq n$, put

$$\alpha_k = \|f_k(x_k(\xi), \varphi_k(x_k(\xi))) - f_k(x_k(\bar{\xi}), \varphi_k(x_k(\bar{\xi})))\|$$

with $\xi, \bar{\xi} \in B_n(\delta\beta_n)$. From (30) it follows that

$$\|(Jx)_m(\xi) - (Jx)_m(\bar{\xi})\| \leq \|\mathcal{A}_{m,n} P_n\| \|\xi - \bar{\xi}\| + \sum_{k=n}^{m-1} \|\mathcal{A}_{m,k+1} P_{k+1}\| \alpha_k \quad (31)$$

for every $m > n$. From (5), (23), (26) and (27) we obtain

$$\begin{aligned} \alpha_k &\leq c (\|x_k(\xi) - x_k(\bar{\xi})\| + \|\varphi_k(x_k(\xi)) - \varphi_k(x_k(\bar{\xi}))\|) \times \\ &\quad \times (\|x_k(\xi)\| + \|\varphi_k(x_k(\xi))\| + \|x_k(\bar{\xi})\| + \|\varphi_k(x_k(\bar{\xi}))\|)^q \\ &\leq 3^{q+1} c \|x_k(\xi) - x_k(\bar{\xi})\| (\|x_k(\xi)\| + \|x_k(\bar{\xi})\|)^q \\ &\leq c(3C)^{q+1} (2\delta)^q \left(\frac{\mu_k}{\mu_{n-1}} \right)^{aq+a} \nu_{n-1}^{\varepsilon(q+1)} \beta_n^q \|\xi - \bar{\xi}\|. \end{aligned} \quad (32)$$

By (14) we get

$$\mu_{n-1}^{-aq} \nu_{n-1}^{\varepsilon(q+1)} \beta_n^q \sum_{k=n}^{+\infty} \mu_k^{aq} \nu_k^\varepsilon = 1 \quad (33)$$

and this together with (32) and (2) imply

$$\begin{aligned}
& \sum_{k=n}^{m-1} \|\mathcal{A}_{m,k+1} P_{k+1}\| \alpha_k \\
& \leq c(3C)^{q+1} D(2\delta)^q \left(\frac{\mu_m}{\mu_{n-1}} \right)^a \|\xi - \bar{\xi}\| \mu_{n-1}^{-aq} \nu_{n-1}^{\varepsilon(q+1)} \beta_n^q \sum_{k=n}^{m-1} \mu_k^{aq} \nu_k^\varepsilon \\
& \leq c(3C)^{q+1} D(2\delta)^q \left(\frac{\mu_m}{\mu_{n-1}} \right)^a \|\xi - \bar{\xi}\|.
\end{aligned}$$

From last estimate, (31) and (2) we have

$$\begin{aligned}
& \|(Jx)_m(\xi) - (Jx)_m(\bar{\xi})\| \\
& \leq D \left(\frac{\mu_m}{\mu_{n-1}} \right)^a \nu_{n-1}^\varepsilon \|\xi - \bar{\xi}\| + c(3C)^{q+1} D(2\delta)^q \left(\frac{\mu_m}{\mu_{n-1}} \right)^a \|\xi - \bar{\xi}\|
\end{aligned}$$

for every $m \geq n$ and every $\xi, \bar{\xi} \in B_n(\delta\beta_n)$. Since $C > D$, choosing δ sufficiently small we obtain

$$\|(Jx)_m(\xi) - (Jx)_m(\bar{\xi})\| \leq C \left(\frac{\mu_m}{\mu_{n-1}} \right)^a \nu_{n-1}^\varepsilon \|\xi - \bar{\xi}\|$$

and this implies the inclusion $J(\mathcal{B}_{n,\delta,\beta}) \subset \mathcal{B}_{n,\delta,\beta}$.

We now show that J is a contraction for the metric induced by (28). Let $x, y \in \mathcal{B}_{n,\delta,\beta}$. Then

$$\begin{aligned}
& \|(Jx)_m(\xi) - (Jy)_m(\xi)\| \\
& \leq \sum_{k=n}^{m-1} \|\mathcal{A}_{m,k+1} P_{k+1}\| \|f_k(x_k(\xi), \varphi_k(x_k(\xi))) - f_k(y_k(\xi), \varphi_k(y_k(\xi)))\| \quad (34)
\end{aligned}$$

for every $m \geq n$ and every $\xi \in B_n(\delta\beta_n)$. By (5), (23), (28) and (27) we have for every $k \geq n$

$$\begin{aligned}
& \|f_k(x_k(\xi), \varphi_k(x_k(\xi))) - f_k(y_k(\xi), \varphi_k(y_k(\xi)))\| \\
& \leq c(\|x_k(\xi) - y_k(\xi)\| + \|\varphi_k(x_k(\xi)) - \varphi_k(y_k(\xi))\|) \times \\
& \quad \times (\|x_k(\xi)\| + \|\varphi_k(x_k(\xi))\| + \|y_k(\xi)\| + \|\varphi_k(y_k(\xi))\|)^q \\
& \leq 3^{q+1} c \|x_k(\xi) - y_k(\xi)\| (\|x_k(\xi)\| + \|y_k(\xi)\|)^q \\
& \leq 3^{q+1} c \left(\frac{\mu_k}{\mu_{n-1}} \right)^a \nu_{n-1}^\varepsilon \|x - y\|'' \|\xi\| (2C\delta)^q \left(\frac{\mu_k}{\mu_{n-1}} \right)^{aq} \nu_{n-1}^{\varepsilon q} \beta_n^q \\
& \leq 3^{q+1} c (2C\delta)^q \left(\frac{\mu_k}{\mu_{n-1}} \right)^{aq+a} \nu_{n-1}^{\varepsilon(q+1)} \beta_n^q \|x - y\|'' \|\xi\|. \quad (35)
\end{aligned}$$

Hence, from (34), (2) and (35) we have

$$\begin{aligned}
& \|(Jx)_m(\xi) - (Jy)_m(\xi)\| \\
& \leq 3^{q+1} c (2C\delta)^q D \left(\frac{\mu_m}{\mu_{n-1}} \right)^a \|\xi\| \|x - y\|'' \mu_{n-1}^{-aq} \nu_{n-1}^{\varepsilon(q+1)} \beta_n^q \sum_{k=n}^{m-1} \mu_k^{aq} \nu_k^\varepsilon \\
& \leq 3^{q+1} c (2C\delta)^q D \left(\frac{\mu_m}{\mu_{n-1}} \right)^a \|\xi\| \|x - y\|''
\end{aligned}$$

for every $m \geq n$ and every $\xi \in B_n(\delta\beta_n)$ and this implies

$$\|Jx - Jy\|'' \leq 3^{q+1}c(2C\delta)^q D \|x - y\|''.$$

Choosing $\delta > 0$ such that $3^{q+1}c(2C\delta)^q D < 1$ it follows that J is a contraction in $\mathcal{B}_{n,\delta,\beta}$. Because $\mathcal{B}_{n,\delta,\beta}$ is complete, by the Banach fixed point theorem, the map J has a unique fixed point x^φ in $\mathcal{B}_{n,\delta,\beta}$, which is thus the desired sequence.

Next we will prove (29). Let $\varphi, \psi \in \mathcal{X}_{\delta,\beta}^*$. From (20) we have

$$\begin{aligned} & \|x_m^\varphi(\xi) - x_m^\psi(\xi)\| \\ & \leq \sum_{k=n}^{m-1} \|\mathcal{A}_{m,k+1} P_{k+1}\| \|f_k(x_k^\varphi(\xi), \varphi_k(x_k^\varphi(\xi))) - f_k(x_k^\psi(\xi), \psi_k(x_k^\psi(\xi)))\| \end{aligned} \quad (36)$$

for every $m \geq n$ and every $\xi \in B_n(\delta\beta_n)$. By (5), (23), (28), (24) and (27) it follows that

$$\begin{aligned} & \|f_k(x_k^\varphi(\xi), \varphi_k(x_k^\varphi(\xi))) - f_k(x_k^\psi(\xi), \psi_k(x_k^\psi(\xi)))\| \\ & \leq c \left(\|x_k^\varphi(\xi) - x_k^\psi(\xi)\| + \|\varphi_k(x_k^\varphi(\xi)) - \psi_k(x_k^\psi(\xi))\| \right) \times \\ & \quad \times \left(\|x_k^\varphi(\xi)\| + \|\varphi_k(x_k^\varphi(\xi))\| + \|x_k^\psi(\xi)\| + \|\psi_k(x_k^\psi(\xi))\| \right)^q \\ & \leq c \left(3\|x_k^\varphi(\xi) - x_k^\psi(\xi)\| + \|\varphi_k(x_k^\psi(\xi)) - \psi_k(x_k^\psi(\xi))\| \right) \left(3\|x_k^\varphi(\xi)\| + 3\|x_k^\psi(\xi)\| \right)^q \\ & \leq c(6C\delta)^q \left(3\|x^\varphi - x^\psi\|'' \left(\frac{\mu_k}{\mu_{n-1}} \right)^a \nu_{n-1}^\varepsilon \|\xi\| + \|\varphi - \psi\|' \|x_k^\psi(\xi)\| \right) \times \\ & \quad \times \left(\frac{\mu_k}{\mu_{n-1}} \right)^{aq} \nu_{n-1}^{\varepsilon q} \beta_n^q \\ & \leq c(6C\delta)^q \left(3\|x^\varphi - x^\psi\|'' + C\|\varphi - \psi\|' \right) \|\xi\| \left(\frac{\mu_k}{\mu_{n-1}} \right)^{aq+a} \nu_{n-1}^{\varepsilon(q+1)} \beta_n^q \end{aligned} \quad (37)$$

for every $k \geq n$. Hence by (36), last inequality, (2) and (33) we get

$$\begin{aligned} & \|x_m^\varphi(\xi) - x_m^\psi(\xi)\| \leq c(6C\delta)^q D \left(3\|x^\varphi - x^\psi\|'' + C\|\varphi - \psi\|' \right) \times \\ & \quad \times \|\xi\| \left(\frac{\mu_m}{\mu_{n-1}} \right)^a \nu_{n-1}^\varepsilon \mu_{n-1}^{-aq} \nu_{n-1}^{\varepsilon q} \beta_n^q \sum_{k=n}^m \mu_k^{aq} \nu_k^\varepsilon \\ & \leq c(6C\delta)^q D \left(3\|x^\varphi - x^\psi\|'' + C\|\varphi - \psi\|' \right) \|\xi\| \left(\frac{\mu_m}{\mu_{n-1}} \right)^a \end{aligned}$$

for every $m \geq n$ and every $\xi \in B_n(\delta\beta_n)$ and this implies

$$\|x^\varphi - x^\psi\|'' \leq c(6C\delta)^q D \nu_{n-1}^{-\varepsilon} \left(3\|x^\varphi - x^\psi\|'' + C\|\varphi - \psi\|' \right).$$

Choosing $\delta > 0$ such that $c(6C\delta)^q D < 1/6$ we have (29). \square

We now represent by $\left(x_{n,k}^\varphi\right)_{k \geq n} \in \mathcal{B}_{n,\delta,\beta}$ the unique sequence given by Lemma 1.

Lemma 2. *Given $\delta > 0$ sufficiently small and $\varphi \in \mathcal{X}_{\delta,\beta}^*$ the following properties hold:*

- 1) If for every $n \in \mathbb{N}$, $m \geq n$ and $\xi \in B_n(\delta\beta_n)$ the identity (21) holds with $x = x^\varphi$, then

$$\varphi_n(\xi) = - \sum_{k=n}^{\infty} \mathcal{A}_{k+1,n}^{-1} Q_{k+1} f_k(x_{n,k}^\varphi(\xi), \varphi_k(x_{n,k}^\varphi(\xi))). \quad (38)$$

for every $n \in \mathbb{N}$ and every $\xi \in B_n(\delta\beta_n)$.

- 2) If for every $n \in \mathbb{N}$ and every $\xi \in B_n(\delta\beta_n)$ the equation (38) holds, then (21) holds with $x = x^\varphi$ for every $n \in \mathbb{N}$, every $m \geq n$ and every $\xi \in B_n(\delta\tilde{\beta}_n/(CK))$.

Proof. First we prove that the series in (38) is convergent. From (3), (6), (23) and (27), we conclude that for every $n \in \mathbb{N}$ and every $\xi \in B_n(\delta\beta_n)$

$$\begin{aligned} & \sum_{k=n}^{\infty} \|\mathcal{A}_{k+1,n}^{-1} Q_{k+1} f_k(x_{n,k}^\varphi(\xi), \varphi_k(x_{n,k}^\varphi(\xi)))\| \\ & \leq \sum_{k=n}^{\infty} \|\mathcal{A}_{k+1,n}^{-1} Q_{k+1}\| \|f_k(x_{n,k}^\varphi(\xi), \varphi_k(x_{n,k}^\varphi(\xi)))\| \\ & \leq \sum_{k=n}^{\infty} D \left(\frac{\mu_k}{\mu_n} \right)^{-b} \nu_k^\varepsilon c \left(\|x_{n,k}^\varphi(\xi)\| + \|\varphi_k(x_{n,k}^\varphi(\xi))\| \right)^{q+1} \\ & \leq cD \sum_{k=n}^{\infty} \left(\frac{\mu_k}{\mu_n} \right)^{-b} \nu_k^\varepsilon \left(3C\delta \left(\frac{\mu_k}{\mu_{n-1}} \right)^a \nu_{n-1}^\varepsilon \beta_n \right)^{q+1} \\ & \leq c(3C\delta)^{q+1} D \mu_n^b \mu_{n-1}^{-aq-a} \nu_{n-1}^{\varepsilon(q+1)} \beta_n^{q+1} \sum_{k=n}^{\infty} \mu_k^{aq+a-b} \nu_k^\varepsilon \\ & \leq c(3C\delta)^{q+1} D \mu_{n-1}^{-aq} \nu_{n-1}^{\varepsilon(q+1)} \beta_n^{q+1} \sum_{k=n}^{\infty} \mu_k^{aq} \nu_k^\varepsilon \\ & \leq c(3C\delta)^{q+1} D \beta_n \end{aligned}$$

and thus the series converges.

Now, let us suppose that (21) holds with $x = x^\varphi$ for every $n \in \mathbb{N}$, every $m \geq n$ and every $\xi \in B_n(\delta\beta_n)$. Then, since $\mathcal{A}_{m,n}^{-1} \mathcal{A}_{m,k+1} = \mathcal{A}_{k+1,n}^{-1}$ for $n \leq k \leq m-1$, equation (21) can be written in the following equivalent form

$$\varphi_n(\xi) = \mathcal{A}_{m,n}^{-1} \varphi_m(x_{n,m}^\varphi(\xi)) - \sum_{k=n}^{m-1} \mathcal{A}_{k+1,n}^{-1} Q_{k+1} f_k(x_{n,k}^\varphi(\xi), \varphi_k(x_{n,k}^\varphi(\xi))). \quad (39)$$

Using (3), (23) and (27), we have

$$\begin{aligned} \|\mathcal{A}_{m,n}^{-1} \varphi_m(x_{n,m}^\varphi(\xi))\| &= \|\mathcal{A}_{m,n}^{-1} Q_m \varphi_m(x_{n,m}^\varphi(\xi))\| \\ &\leq 2D \left(\frac{\mu_{m-1}}{\mu_n} \right)^{-b} \nu_{m-1}^\varepsilon \|x_{n,m}^\varphi(\xi)\| \\ &\leq 2D \left(\frac{\mu_{m-1}}{\mu_n} \right)^{-b} \nu_{m-1}^\varepsilon C\delta \left(\frac{\mu_m}{\mu_{n-1}} \right)^a \nu_{n-1}^\varepsilon \beta_n \\ &\leq 2CD\delta \mu_m^a \mu_{m-1}^{-b} \nu_{m-1}^\varepsilon \mu_n^b \mu_{n-1}^{-a} \nu_{n-1}^\varepsilon \beta_n \end{aligned}$$

and by (12) this converge to zero when $m \rightarrow \infty$. Hence, letting $m \rightarrow \infty$ in (39) we obtain the identity (38) for every $n \in \mathbb{N}$ and every $\xi \in B_n(\delta\beta_n)$.

We now assume that for every $n \in \mathbb{N}$, $m \geq n$ and $\xi \in B_n(\delta\beta_n)$ the identity (38) holds. If $\xi \in B_n(\delta\tilde{\beta}_n/(CK))$, then by (15) we get

$$\|x_{n,m}(\xi)\| \leq C \left(\frac{\mu_m}{\mu_{n-1}} \right)^a \nu_{n-1}^{\varepsilon} \frac{\delta}{CK} \tilde{\beta}_n = \frac{\delta}{K} \frac{\mu_m^a \beta_m^{-1}}{\mu_{n-1}^a \beta_n^{-1}} \beta_m \leq \delta\beta_m. \quad (40)$$

Therefore

$$\mathcal{A}_{m,n}\varphi_n(\xi) = - \sum_{k=n}^{\infty} \mathcal{A}_{m,n} \mathcal{A}_{k+1,n}^{-1} Q_{k+1} f_k(x_{n,k}^{\varphi}(\xi), \varphi_k(x_{n,k}^{\varphi}(\xi))),$$

and thus it follows from (38) and the uniqueness of the sequences x^{φ} that

$$\begin{aligned} \mathcal{A}_{m,n}\varphi_n(\xi) &+ \sum_{k=n}^{m-1} \mathcal{A}_{m,k+1} Q_{k+1} f_k(x_{n,k}^{\varphi}(\xi), \varphi_k(x_{n,k}^{\varphi}(\xi))) \\ &= - \sum_{k=m}^{\infty} \mathcal{A}_{k+1,m}^{-1} Q_{k+1} f_k(x_{n,k}^{\varphi}(\xi), \varphi_k(x_{n,k}^{\varphi}(\xi))) \\ &= - \sum_{k=m}^{\infty} \mathcal{A}_{k+1,m}^{-1} Q_{k+1} f_k(x_{m,k}^{\varphi}(x_{n,m}^{\varphi}(\xi)), \varphi_k(x_{m,k}^{\varphi}(x_{n,m}^{\varphi}(\xi)))) \\ &= \varphi_m(x_{n,m}^{\varphi}(\xi)) \end{aligned}$$

for every $n \in \mathbb{N}$, every $m \geq n$ and every $\xi \in B_n(\delta\beta_n/(CK))$. This proves the lemma. \square

Lemma 3. *Given $\delta > 0$ sufficiently small there is a unique $\varphi \in \mathcal{X}_{\delta,\beta}^*$ such that*

$$\varphi_n(\xi) = - \sum_{k=n}^{\infty} \mathcal{A}_{k+1,n}^{-1} Q_{k+1} f_k(x_k^{\varphi}(\xi), \varphi_k(x_k^{\varphi}(\xi)))$$

for every $n \in \mathbb{N}$ and every $\xi \in B_n(\delta\beta_n)$.

Proof. We consider the operator Φ defined for each $\varphi \in \mathcal{X}_{\delta,\beta}^*$ by

$$(\Phi\varphi)_n(\xi) = \begin{cases} - \sum_{k=n}^{\infty} \mathcal{A}_{k+1,n}^{-1} Q_{k+1} f_k(x_k^{\varphi}(\xi), \varphi_k(x_k^{\varphi}(\xi))) & \text{if } \xi \in B_n(\delta\beta_n), \\ (\Phi\varphi)_n(\delta\beta_n\xi/\|\xi\|) & \text{if } \xi \notin B_n(\delta\beta_n), \end{cases} \quad (41)$$

where $x^{\varphi} = (x_k^{\varphi})_{k \geq n} \in \mathcal{B}_{n,\delta,\beta}$ is the unique sequence given by Lemma 1. It follows from (4), (25), (9) and (41) that $(\Phi\varphi)_n(0) = 0$ for each $n \in \mathbb{N}$.

Furthermore, given $n \in \mathbb{N}$ and $\xi, \bar{\xi} \in B_n(\delta\beta_n)$, by (3), (32) and (33) we have

$$\begin{aligned} &\|(\Phi\varphi)_n(\xi) - (\Phi\varphi)_n(\bar{\xi})\| \\ &\leq \sum_{k=n}^{\infty} \|\mathcal{A}_{k+1,n}^{-1} Q_{k+1}\| \cdot \|f_k(x_k^{\varphi}(\xi), \varphi_k(x_k^{\varphi}(\xi))) - f_k(x_k^{\varphi}(\bar{\xi}), \varphi_k(x_k^{\varphi}(\bar{\xi})))\| \\ &\leq c(3C)^{q+1} D (2\delta)^q \|\xi - \bar{\xi}\| \mu_{n-1}^{-aq-a} \mu_n^b \nu_{n-1}^{\varepsilon(q+1)} \beta_n^q \sum_{k=n}^{\infty} \mu_k^{aq+a-b} \nu_k^{\varepsilon} \\ &\leq c(3C)^{q+1} D (2\delta)^q \|\xi - \bar{\xi}\| \mu_{n-1}^{-aq} \nu_{n-1}^{\varepsilon(q+1)} \beta_n^q \sum_{k=n}^{\infty} \mu_k^{aq} \nu_k^{\varepsilon} \\ &= c(3C)^{q+1} D (2\delta)^q \|\xi - \bar{\xi}\|. \end{aligned}$$

Hence, choosing $\delta > 0$ (independently of φ , n and ξ) such that $c(3C)^{q+1}D(2\delta)^q \leq 1$ we have

$$\|(\Phi\varphi)_n(\xi) - (\Phi\varphi)_n(\bar{\xi})\| \leq \|\xi - \bar{\xi}\|.$$

Therefore $\Phi(\mathcal{X}_{\delta,\beta}^*) \subset \mathcal{X}_{\delta,\beta}^*$.

We now show that Φ is a contraction. Given $\varphi, \psi \in \mathcal{X}_{\delta,\beta}^*$ and $n \in \mathbb{N}$, let x^φ and x^ψ be the unique sequences given by Lemma 1 respectively for φ and ψ . By (3), (37), (29) and (33) we have

$$\begin{aligned} & \|(\Phi\varphi)_n(\xi) - (\Phi\psi)_n(\xi)\| \\ & \leq \sum_{k=n}^{\infty} \|\mathcal{A}_{k+1,n}^{-1} Q_{k+1}\| \|f_k(x_k^\varphi(\xi), \varphi_k(x_k^\varphi(\xi))) - f_k(x_k^\psi(\xi), \varphi_k(x_k^\psi(\xi)))\| \\ & \leq cD(6C\delta)^q (3\|x^\varphi - x^\psi\|'' + C\|\varphi - \psi\|') \|\xi\| \mu_{n-1}^{-aq-a} \mu_n^b \nu_{n-1}^{\varepsilon(q+1)} \beta_n^q \sum_{k=n}^{\infty} \mu_k^{aq+a-b} \nu_k^\varepsilon \\ & \leq 2cC^{q+1}D(6\delta)^q \|\xi\| \|\varphi - \psi\|' \mu_{n-1}^{-aq} \nu_{n-1}^{\varepsilon(q+1)} \beta_n^q \sum_{k=n}^{\infty} \mu_k^{aq} \nu_k^\varepsilon \\ & = 2cC^{q+1}D(6\delta)^q \|\xi\| \|\varphi - \psi\|' \end{aligned}$$

for every $n \in \mathbb{N}$ and every $\xi \in B_n(\delta\beta_n)$ and this implies

$$\|\Phi\varphi - \Phi\psi\|' \leq 2cC^{q+1}D(6\delta)^q \|\varphi - \psi\|'.$$

Choosing $\delta > 0$ such that $2cC^{q+1}D(6\delta)^q < 1$ it follows that Φ is a contraction in $\mathcal{X}_{\delta,\beta}^*$. Therefore the map Φ has a unique fixed point φ in $\mathcal{X}_{\delta,\beta}^*$ that is the desired sequence. \square

We are now in conditions to prove Theorem 1.

Proof of Theorem 1. By Lemma 1, for each $\varphi \in \mathcal{X}_{\delta,\beta}^*$ there is a unique sequence $x^\varphi \in \mathcal{B}_{n,\delta,\beta}$ satisfying (20). It remains to show that there is a φ and a corresponding x^φ that satisfy (21). By Lemma 2, this is equivalent to solve (38). Finally, by Lemma 3, there is a unique solution of (38). This establishes the existence of the stable manifolds for $\delta > 0$ sufficiently small. Moreover, for each $n \in \mathbb{N}$, $m \geq n$ and $\xi, \bar{\xi} \in B_n(\delta\beta_n/(CK))$ it follows from (40) and (10) that

$$\begin{aligned} & \|\mathcal{F}_{m,n}(\xi, \varphi_n(\xi)) - \mathcal{F}_{m,n}(\xi, \varphi_n(\bar{\xi}))\| \\ & \leq \|x_m(\xi) - x_m(\bar{\xi})\| + \|\varphi_m(x_m(\xi)) - \varphi_m(x_m(\bar{\xi}))\| \\ & \leq 2\|x_m(\xi) - x_m(\bar{\xi})\| \\ & \leq 2C \left(\frac{\mu_m}{\mu_{n-1}} \right)^a \nu_{n-1}^\varepsilon \|\xi - \bar{\xi}\|. \end{aligned}$$

Hence we obtain (17) and the theorem is proved. \square

5. BEHAVIOR UNDER PERTURBATIONS

In this section we assume that equation (1) admits a (μ, ν) -dichotomy for some $D \geq 1$, $a < 0 \leq b$ and $\varepsilon \geq 0$. Given $c > 0$ and $q > 1$, let $\mathcal{P}_{c,q}$ be the class of all

sequences of function $f = (f_n)_{n \in \mathbb{N}}$ such that $f_n: X \rightarrow X$ and verify conditions (4) and (5) with the given c and q . In $\mathcal{P}_{c,q}$ we can define a metric by

$$\|f - \bar{f}\|''' = \sup \left\{ \frac{\|f_n(u) - \bar{f}_n(u)\|}{\|u\|^{q+1}} : n \in \mathbb{N}, u \in X \setminus \{0\} \right\}, \quad (42)$$

for every $f = (f_n)_{n \in \mathbb{N}}, \bar{f} = (\bar{f}_n)_{n \in \mathbb{N}} \in \mathcal{P}_{c,q}$.

The purpose of this section is to see how the manifolds in Theorem 1 vary with the perturbations. To do this we consider two sequence of perturbations $f, \bar{f} \in \mathcal{P}_{c,q}$ and the functions φ and $\bar{\varphi}$ given by Theorem 1 when we perturb equation (8) with f and \bar{f} , respectively, and we compare the distance between φ and $\bar{\varphi}$ in the metric given by (22) with the distance between f and \bar{f} in the metric given by (42).

Theorem 2. *Let $c > 0$ and $q > 1$. Suppose that equation (1) admits a (μ, ν) -dichotomy for some $D \geq 1$, $a < 0 \leq b$ and $\varepsilon > 0$ and that the hypothesis of Theorem 1 are satisfied. Then, choosing for $\delta > 0$ sufficiently small, we have*

$$\|\varphi - \bar{\varphi}\|' \leq \|f - \bar{f}\|'''$$

for every $f, \bar{f} \in \mathcal{P}_{c,q}$, where $\varphi, \bar{\varphi} \in \mathcal{X}_{\delta,\beta}$ are the functions given by Theorem 1 for the same constant $C > D$ corresponding to the perturbations f and \bar{f} , respectively.

Proof. Let $n \in \mathbb{N}$ and $\xi \in B_n(\delta\beta_n)$. From (38), putting for every $k \geq n$

$$\gamma_k := \|f_k(x_k^\varphi(\xi), \varphi_k(x_k^\varphi(\xi))) - \bar{f}_k(x_k^{\bar{\varphi}}(\xi), \bar{\varphi}_k(x_k^{\bar{\varphi}}(\xi)))\|,$$

we obtain

$$\|\varphi_n(\xi) - \bar{\varphi}_n(\xi)\| \leq \sum_{k=n}^{+\infty} \|\mathcal{A}_{k+1,n}^{-1} Q_{k+1}\| \gamma_k, \quad (43)$$

where $x^\varphi, x^{\bar{\varphi}} \in \mathcal{B}_{n,\delta,\beta}$ are the sequences of functions given by Lemma 1 associated with (f, φ) and $(\bar{f}, \bar{\varphi})$, respectively. By (42), (5), (23), (24), (27) and (28) we have for $k \geq n$

$$\begin{aligned} \gamma_k &\leq \|f_k(x_k^\varphi(\xi), \varphi_k(x_k^\varphi(\xi))) - \bar{f}_k(x_k^\varphi(\xi), \varphi_k(x_k^\varphi(\xi)))\| \\ &\quad + \|\bar{f}_k(x_k^\varphi(\xi), \varphi_k(x_k^\varphi(\xi))) - \bar{f}_k(x_k^{\bar{\varphi}}(\xi), \bar{\varphi}_k(x_k^{\bar{\varphi}}(\xi)))\| \\ &\leq 3^{q+1} \|f - \bar{f}\|''' \|x_k^\varphi(\xi)\|^{q+1} \\ &\quad + 3^{q+1} c \|x_k^\varphi(\xi) - x_k^{\bar{\varphi}}(\xi)\| (\|x_k^\varphi(\xi)\| + \|x_k^{\bar{\varphi}}(\xi)\|)^q \\ &\quad + 3^q c \|\varphi - \bar{\varphi}\|' \|x_k^{\bar{\varphi}}(\xi)\| (\|x_k^\varphi(\xi)\| + \|x_k^{\bar{\varphi}}(\xi)\|)^q \\ &\leq 3^{q+1} C^{q+1} \delta^q \|f - \bar{f}\|''' \|\xi\| \left(\frac{\mu_k}{\mu_{n-1}} \right)^{aq+a} \nu_{n-1}^{\varepsilon(q+1)} \beta_n^q \\ &\quad + 2^q 3^{q+1} c C^q \delta^q \|x^\varphi - x^{\bar{\varphi}}\|'' \|\xi\| \left(\frac{\mu_k}{\mu_{n-1}} \right)^{aq+a} \nu_{n-1}^{\varepsilon(q+1)} \beta_n^q \\ &\quad + 2^q 3^q c C^{q+1} \delta^q \|\varphi - \bar{\varphi}\|' \|\xi\| \left(\frac{\mu_k}{\mu_{n-1}} \right)^{aq+a} \nu_{n-1}^{\varepsilon(q+1)} \beta_n^q \end{aligned} \quad (44)$$

and using (33), the last estimate, (43), we get

$$\begin{aligned}
\|\varphi_n(\xi) - \bar{\varphi}_n(\xi)\| &\leq 3^{q+1}C^{q+1}D\delta^q\|f - \bar{f}\|'''\|\xi\|\mu_{n-1}^{-aq}\nu_{n-1}^{\varepsilon(q+1)}\beta_n^q \sum_{k=n}^{+\infty} \mu_k^{aq}\nu_k^\varepsilon \\
&\quad + 2^q3^{q+1}cC^qD\delta^q\|x^\varphi - x^{\bar{\varphi}}\|''\|\xi\|\mu_{n-1}^{-aq}\nu_{n-1}^{\varepsilon(q+1)}\beta_n^q \sum_{k=n}^{+\infty} \mu_k^{aq}\nu_k^\varepsilon \\
&\quad + 2^q3^qcC^{q+1}D\delta^q\|\varphi - \bar{\varphi}\|'\|\xi\|\mu_{n-1}^{-aq}\nu_{n-1}^{\varepsilon(q+1)}\beta_n^q \sum_{k=n}^{+\infty} \mu_k^{aq}\nu_k^\varepsilon \\
&\leq 3^{q+1}C^{q+1}D\delta^q\|f - \bar{f}\|'''\|\xi\| + 2^q3^{q+1}cC^qD\delta^q\|x^\varphi - x^{\bar{\varphi}}\|''\|\xi\| \\
&\quad + 2^q3^qcC^{q+1}D\delta^q\|\varphi - \bar{\varphi}\|'\|\xi\|.
\end{aligned} \tag{45}$$

Now, we will estimate $\|x^\varphi - x^{\bar{\varphi}}\|''$. By (20), (44) and (2) we obtain for every $m \geq n$ and every $\xi \in B_n(\delta\beta_n)$

$$\begin{aligned}
&\left(\frac{\mu_m}{\mu_{n-1}}\right)^{-a} \nu_{n-1}^{-\varepsilon} \|x_m^\varphi(\xi) - x_m^{\bar{\varphi}}(\xi)\| \\
&\leq \left(\frac{\mu_m}{\mu_{n-1}}\right)^{-a} \nu_{n-1}^{-\varepsilon} \sum_{k=n}^{m-1} \|\mathcal{A}_{m,k+1}P_{k+1}\| \gamma_k \\
&\leq 3^{q+1}C^{q+1}D\delta^q\|f - \bar{f}\|'''\|\xi\|\mu_{n-1}^{-aq}\nu_{n-1}^{\varepsilon q}\beta_n^q \sum_{k=n}^{m-1} \mu_k^{aq}\nu_k^\varepsilon \\
&\quad + 2^q3^{q+1}cC^qD\delta^q\|x^\varphi - x^{\bar{\varphi}}\|''\|\xi\|\mu_{n-1}^{-aq}\nu_{n-1}^{\varepsilon q}\beta_n^q \sum_{k=n}^{m-1} \mu_k^{aq}\nu_k^\varepsilon \\
&\quad + 2^q3^qcC^{q+1}D\delta^q\|\varphi - \bar{\varphi}\|'\|\xi\|\mu_{n-1}^{-aq}\nu_{n-1}^{\varepsilon q}\beta_n^q \sum_{k=n}^{m-1} \mu_k^{aq}\nu_k^\varepsilon \\
&\leq 3^{q+1}C^{q+1}D\delta^q\|f - \bar{f}\|'''\|\xi\| + 2^q3^{q+1}cC^qD\delta^q\|x^\varphi - x^{\bar{\varphi}}\|''\|\xi\| \\
&\quad + 2^q3^qcC^{q+1}D\delta^q\|\varphi - \bar{\varphi}\|'\|\xi\|
\end{aligned}$$

and this implies

$$\begin{aligned}
&\|x^\varphi - x^{\bar{\varphi}}\|'' \\
&\leq 3^{q+1}C^{q+1}D\delta^q\|f - \bar{f}\|''' + 2^q3^{q+1}cC^qD\delta^q\|x^\varphi - x^{\bar{\varphi}}\|'' + 2^q3^qcC^{q+1}D\delta^q\|\varphi - \bar{\varphi}\|'.
\end{aligned}$$

Thus, for $\delta > 0$ such that $2^q3^{q+1}cC^qD\delta^q < 1/2$ we have

$$\|x^\varphi - x^{\bar{\varphi}}\|'' \leq 2 \cdot 3^{q+1}C^{q+1}D\delta^q\|f - \bar{f}\|''' + 2^{q+1}3^qcC^{q+1}D\delta^q\|\varphi - \bar{\varphi}\|'.$$

It follows from the last estimate, (45) and $2^q3^{q+1}cC^qD\delta^q < 1/2$ that

$$\|\varphi_n(\xi) - \bar{\varphi}_n(\xi)\| \leq 2 \cdot 3^{q+1}C^{q+1}D\delta^q\|f - \bar{f}\|'''\|\xi\| + 2^{q+1}3^qcC^{q+1}D\delta^q\|\varphi - \bar{\varphi}\|'\|\xi\|$$

for every $n \in \mathbb{N}$ and every $\xi \in B_n(\delta\beta_n)$. Hence we get

$$\|\varphi - \bar{\varphi}\|' \leq 2 \cdot 3^{q+1}C^{q+1}D\delta^q\|f - \bar{f}\|''' + 2^{q+1}3^qcC^{q+1}D\delta^q\|\varphi - \bar{\varphi}\|'$$

and choosing $\delta > 0$ such that $2^{q+1}3^qcC^{q+1}D\delta^q < 1/2$ we obtain

$$\|\varphi - \bar{\varphi}\|' \leq 4 \cdot 3^{q+1}C^{q+1}D\delta^q\|f - \bar{f}\|'''.$$

To finish the proof we have to choose $\delta > 0$ such that $4 \cdot 3^{q+1} C^{q+1} D \delta^q \leq 1$. \square

ACKNOWLEDGMENTS

This work was supported by Centro de Matemática da Universidade da Beira Interior.

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